Extinction time for a random walk in a random environment

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Abstract

We consider a random walk with death in $[-N, N]$ moving in a time dependent environment. The environment is a system of particles which describes a current flux from N to $-N$. Its evolution is influenced by the presence of the random walk and in turns it affects the jump rates of the random walk in a neighborhood of the endpoints, determining also the rate for the random walk to die. We prove an upper bound (uniform in N) for the probability of extinction by time t which goes as $c\exp\{-bN^{-2}t\}$, c and b positive constants.

1 Introduction

We consider a random walk on the discrete interval $\Lambda_N := [-N, N]$ of $\mathbb{Z}$ which eventually dies by jumping to a final state $\emptyset$ (where it stays thereafter). Let $z \in \Lambda_N \cup \emptyset$ denote the state of the random walk, of which we say to be *alive* when $z \in \Lambda_N$ and *dead* when $z = \emptyset$. When z is alive and $|z| \le N - 2$ it moves as a simple random walk: after an exponential time of mean 1 it jumps to its right or left neighbor with probability 1/2. When $z \in I$, $I = I_+ \cup I_-$, $I_+ = \{N-1, N\}$, $I_- = \{-N, -N+1\}$ then, besides moving, the walk z may also die. The jump and death rates depend on the environment.

The environment is a particles configuration η on $\Lambda_N \setminus \{z\}$, z the state of the random walk (i.e. if $z = \emptyset$ then $\eta \in \{0,1\}^{\Lambda_N}$, otherwise $\eta \in \{0,1\}^{\Lambda_N \setminus z}$). The evolution of the

environment is influenced by the motion of the random walk: it consists of jumps of the particles (as second class symmetric exclusion particles with z being first class) plus birth-death events localized in I. The precise formulation is given in the next section. We just mention here that the birth-deaths events are "rare" as their intensity is proportional to $1/N$ and we are interested in the case of large N.

When $z = \emptyset$ the environment evolves as in [1] with $K = 2$ there. Namely, it is the simple symmetric exclusion process (SSEP, see [6], [7]) in Λ_N plus injection of particles into I_+ and removal from I_-, hereby referred as DPTV: at rate $j/(2N)$ one tries to inject a particle at the rightmost empty site in I_+ and at the same rate there is an attempt to remove the leftmost particle in I_-, the corresponding action being aborted if I_+ is full or I_- is empty. Thus, when the random walk is dead the η process describes a flux of particles from right to left and it models how currents can be induced by "current reservoirs", represented here by the injection and removal processes at I_+ and respectively I_-.

The presence of the random walk changes the picture and the purpose of this paper is to study how long does such an influence persist: we shall prove that the survival probability of the random walk decreases exponentially in time, being bounded above by $c\exp\{-bN^{-2}t\}$, $c, b > 0$ independent of t and N. In a companion paper [4] we use the techniques and results developed here to bound the extinction time in the case of several random walks. These random walks correspond to the positions of discrepancies between two configurations that evolve according to the DPTV process mentioned before. By stochastic inequalities the result yields a lower bound of the form bN^{-2} for the spectral gap in this process, which is the motivation for our study here.

2 Model and results

The evolution of (z, η) (random walk plus environment) is a Markov process defined by a generator L which is the sum of the generators defined below, in (2.1)–(2.8). Letting the value $\eta(x) = 1$ ($\eta(x) = 0$) indicate the presence (absence) of a particle at x, we may for convenience always take $\eta \in \{0, 1\}^{\Lambda_N}$ by requesting that $\eta(z) = 0$ whenever $z \neq \emptyset$.

We first suppose $z \neq \emptyset$ and write:

$$L^0_{\mathrm{env}} f(z, \eta) = \frac{1}{2}\{\sum_{x=-N}^{z-2} + \sum_{x=z+1}^{N-1}\}[f(z, \eta^{(x,x+1)}) - f(z, \eta)] \tag{2.1}$$

$$L^0_{\mathrm{z}} f(z, \eta) = \frac{1}{2}\{\mathbf{1}_{z<N}[f(z+1, \eta^{(z,z+1)}) - f(z, \eta)] + \mathbf{1}_{z>-N}[f(z-1, \eta^{(z-1,z)}) - f(z, \eta)]\}, \tag{2.2}$$

where $\eta^{(x,x+1)}$ is obtained from η by interchanging the occupation values at x and $x + 1$, and $\mathbf{1}_{z\in A}$ refers to the indicator function.

Denoting by $\eta^{(+,x)}$ ($\eta^{(-,x)}$) the configuration which has the value 1 (0 resp.) at x and

coincides with η otherwise,

$$
\begin{aligned}
L^+_{\rm env} f(z,\eta) &= \frac{j}{2N}\Big\{\mathbf{1}_{z<N}(1-\eta(N))[f(z,\eta^{(+,N)})-f(z,\eta)] \\
&+ \mathbf{1}_{z<N-1}(1-\eta(N-1))\eta(N)[f(z,\eta^{(+,N-1)})-f(z,\eta)]\Big\}
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
L^-_{\rm env} f(z,\eta) &= \frac{j}{2N}\Big\{\mathbf{1}_{z>-N}\eta(-N))[f(z,\eta^{(-,-N)})-f(z,\eta)] \\
&+ \mathbf{1}_{z>-N+1}\eta(-N+1))(1-\eta(-N))[f(z,\eta^{(-,-N+1)})-f(z,\eta)]\Big\}
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
L^+_{\rm death} f(z,\eta) &= \frac{j}{2N}\Big\{\mathbf{1}_{z=N}\eta(N-1)[f(\emptyset,\eta^{(+,N)})-f(z,\eta)] \\
&+ \mathbf{1}_{z=N-1}\eta(N)[f(\emptyset,\eta^{(+,N-1)})-f(z,\eta)]\Big\}
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
L^-_{\rm death} f(z,\eta) &= \frac{j}{2N}\Big\{\mathbf{1}_{z=-N}(1-\eta(-N+1)[f(\emptyset,\eta^{(-,-N)})-f(z,\eta)] \\
&+ \mathbf{1}_{z=-N+1}(1-\eta(-N))[f(\emptyset,\eta^{(-,-N+1)})-f(z,\eta)]\Big\}
\end{aligned}
\tag{2.6}
$$

$$
L^+_{\rm z} f(z,\eta) = \frac{j}{2N}\mathbf{1}_{z=N}(1-\eta(N-1))[f(N-1,\eta^{(+,N)})-f(z,\eta)]
\tag{2.7}
$$

$$
L^-_{\rm z} f(z,\eta) = \frac{j}{2N}\mathbf{1}_{z=-N}\eta(-N+1)[f(-N+1,\eta^{(-,-N)})-f(z,\eta)].
\tag{2.8}
$$

When $z=\emptyset$ the generator is the sum of those in (2.1), (2.3) and (2.4) after replacing the indicator functions by 1 and putting $z=\emptyset$. It is the one considered in [1] in the special case when the sets $I_\pm$ consist of only two sites.

Denote by $(z_t,\eta_t)_{t\ge 0}$ the Markov process with the above generator and by $P_{z,\eta}$ its law starting from (z,η). We now state the main result to be proven in the next sections.

Theorem 1. *There are c and b positive and independent of N so that for any initial datum (z_0,η_0), $z_0\ne\emptyset$ and any $t>0$*

$$
P_{z_0,\eta_0}\big[z_t\ne\emptyset\big]\le ce^{-bN^{-2}t}.
\tag{2.9}
$$

3 The auxiliary process

It will be useful to consider an auxiliary process $(\tilde{z}_t)_{t\geq 0}$. This will be a time-dependent Markov process whose jump intensities at time t are obtained by averaging those of the original process over the environment conditioned on the state of the random walk at that time, the explicit expression of the time dependent generator $\mathcal{L}_t$ is given below in (3.6) after introducing some definitions and notation. We fix hereafter arbitrarily the initial condition (z_0, η_0), $z_0 \neq \emptyset$ and denote by $\tilde{P}_{z_0}$ and $\tilde{E}_{z_0}$ law and expectation of the auxiliary process. We shall prove that for any bounded measurable function $\phi(z, \eta) = f(z)$:

$$E_{z_0,\eta_0}\big[\phi(z_t, \eta_t)\big] = \tilde{E}_{z_0}\big[f(\tilde{z}_t)\big] \tag{3.1}$$

By taking $f(z) = \mathbf{1}_{z\neq\emptyset}$, (3.1) shows that the distributions of the extinction time for the true and the auxiliary processes are the same.
The proof of (3.1) follows from the equality

$$\frac{d}{dt}E_{z_0,\eta_0}\big[\phi(z_t, \eta_t)\big] = E_{z_0,\eta_0}\big[\mathcal{L}_t f(z_t)\big], \tag{3.2}$$

which we shall prove next.

We obviously have $L^{\pm}_{\rm env}\phi = 0$ and, for $z \neq \emptyset$, $L^0_{\rm z}\phi = \mathcal{L}^0 f$ with $\mathcal{L}^0$ the generator of the simple random walk on $[-N, N]$ with jumps outside $[-N, N]$ suppressed (as in the definition of $L^0_{\rm z}$). Recalling (2.5)–(2.6)

$$\begin{aligned}
L^+_{\rm death}\phi &= \frac{j}{2N}\Big\{\mathbf{1}_{z=N}\eta(N-1)[f(\emptyset) - f(N)] + \mathbf{1}_{z=N-1}\eta(N)[f(\emptyset) - f(N-1)]\Big\}\\
L^-_{\rm death}\phi &= \frac{j}{2N}\Big\{\mathbf{1}_{z=-N}(1-\eta(-N+1))[f(\emptyset) - f(-N)]\\
&+ \mathbf{1}_{z=-N+1}(1-\eta(-N))[f(\emptyset) - f(-N+1)]\Big\}
\end{aligned}$$

By (2.7) and (2.8):

$$\begin{aligned}
L^+_{\rm z}\phi &= \frac{j}{2N}\mathbf{1}_{z=N}(1-\eta(N-1))[f(N-1) - f(N)]\\
L^-_{\rm z}\phi &= \frac{j}{2N}\mathbf{1}_{z=-N}\eta(-N+1)[f(-N+1) - f(-N)].
\end{aligned}$$

Thus, we define

$$\begin{aligned}
d(N,t) &= \frac{j}{2N}E_{z_0,\eta_0}[\eta_t(N-1) \mid z_t = N],\\
d(N-1,t) &= \frac{j}{2N}E_{z_0,\eta_0}[\eta_t(N) \mid z_t = N-1]\\
d(-N,t) &= \frac{j}{2N}E_{z_0,\eta_0}[(1-\eta_t(-N+1)) \mid z_t = -N],\\
d(-N+1,t) &= \frac{j}{2N}E_{z_0,\eta_0}[(1-\eta_t(-N)) \mid z_t = -N+1]
\end{aligned} \tag{3.3}$$

set $d(z,t)=0$ if $|z|<N-1$, and let

$$\begin{aligned} a(N,t) &= \frac{j}{2N} E_{z_0,\eta_0}[(1-\eta_t(N-1)) \mid z_t = N], \\ a(-N,t) &= \frac{j}{2N} E_{z_0,\eta_0}[\eta_t(-N+1) \mid z_t = -N]. \end{aligned} \tag{3.4}$$

Given $t \ge 0$ define

$$\begin{aligned} \mathcal{L}^a_t f(z) &= \mathcal{L}^0 f(z) + \mathbf{1}_{z=N} a(N,t)[f(N-1)-f(N)] \\ &+ \mathbf{1}_{z=-N} a(-N,t)[f(-N+1)-f(-N)], \end{aligned} \tag{3.5}$$

and

$$\mathcal{L}_t f(z) = \mathcal{L}^a_t f(z) + d(z,t)[f(\emptyset)-f(z)], \tag{3.6}$$

so that we get (3.2) and hence (3.1) at once.

The auxiliary process $\tilde{z}_t$ is thus the Markov process with time dependent generator $\mathcal{L}_t$. It is a simple random walk with extra jumps $N \to N-1$ and $-N \to -N+1$ which occur with intensities $a(\pm N,t)$ and death rates $(z \to \emptyset)$ given by $d(z,t)$. Calling $\mathcal{P}_{z_0}$ the law of the process $\tilde{z}_t$ with time dependent generator $\mathcal{L}^a_t$ (same fixed η_0 and the same initial condition z_0) and denoting by $\mathcal{E}_{z_0}$ the corresponding expectation, it is not hard to see that

$$\begin{aligned} P_{z_0,\eta_0}\big[z_t \neq \emptyset\big] &= \tilde{P}_{z_0}\big[\tilde{z}_t \neq \emptyset\big] = \mathcal{E}_{z_0}\Big[\exp\{-\int_0^t d(\tilde{z}_s,s)\,ds\}\Big] \\ &\le \mathcal{E}_{z_0}\Big[\exp\{-\int_0^t d(N,s)\mathbf{1}_{\tilde{z}_s=N}\,ds\}\Big] \end{aligned} \tag{3.7}$$

the last inequality is not really necessary, brings some loss, but is just to simplify notation.

The proof of Theorem 1 follows from (3.7) and the following two statements which will be proved in the next sections.

- There are $\delta^* > 0$ and $\kappa > 0$ so that for all $t \ge T_2 = \kappa N^2$:

$$d(N,t) \ge \frac{j\delta^*}{N} \tag{3.8}$$

- There are c and $b>0$ so that calling $T^*(t)$ the total time spent at N by $\tilde{z}_s, 0 \le t$:

$$\mathcal{E}_{z_0}\Big[e^{-j\delta^* N^{-1} T^*(t)}\Big] \le c e^{-bN^{-2}t}, \qquad t \ge T_2 = \kappa N^2 \tag{3.9}$$

4 Proof of (3.9).

Throughout the rest of the paper we shall write $\epsilon \equiv N^{-1}$. With the notation introduced above and writing $\mathcal{E}_{t,\tilde z}$ for the conditional distribution (under $\mathcal{P}_{z_0}$) of $(\tilde z_s, s \ge t)$ given $\tilde z_t = \tilde z$, (3.9) will be consequence of the following statement. Given any $\delta > 0$ there is $p < 1$ so that uniformly in ϵ and for all non negative integers n:

$$\mathcal{E}_{t_n,\tilde z_{t_n}}[e^{-X}] \le p, \quad X := \epsilon\delta \int_{t_n}^{t_{n+1}} \mathbf{1}_{\tilde z_s = N} ds, \quad t_n = 2\epsilon^{-2} n. \tag{4.1}$$

The key point in proving (4.1) is:

Lemma 2. *For any $0 < c_- < c$ there is $p < 1$ (as given in (4.4) below) so that the following holds. Let (Ω, μ) be a probability space, E the expectation and $\mathcal{F}$ the set of all measurable functions $f \ge 0$ such that $E[f] \ge c_-$ and $E[f^2] \le c^2$. Then $E[e^{-f}] \le p$ for any $f \in \mathcal{F}$.*

Proof. Let $f \in \mathcal{F}$, $\zeta := c_-/2$, $\gamma := \mu[f > \zeta]$. Then

$$c_- \le E[f] = E[f; f \le \zeta] + E[f; f > \zeta] \le \zeta(1-\gamma) + c\gamma^{1/2} \tag{4.2}$$

Call $a = \gamma^{1/2}$, then (4.2) yields $\zeta(1-a^2) + ca - c_- \ge 0$, so that $a_- < a < a_+$ where $a_\pm$ are the roots of the corresponding equation with equality:

$$\zeta a^2 - ca + c_- - \zeta = 0, \quad \text{i.e. } 2\zeta a = c \pm \sqrt{c^2 - 4\zeta(c_- - \zeta)} = c \pm \sqrt{c^2 - c_-^2}.$$

Thus

$$2\zeta a_- = c - c\sqrt{1 - \frac{c_-^2}{c^2}} \ge c - c\Big(1 - \frac{1}{2}\frac{c_-^2}{c^2}\Big) = \frac{c_-^2}{2c}$$

so that (since $\mu[f > \zeta] = a^2$ and $a \ge a_-$)

$$\mu[f > \zeta] \ge \Big(\frac{c_-}{2c}\Big)^2 \tag{4.3}$$

and

$$\begin{aligned} E[e^{-f}] &\le e^{-\zeta}\mu[f > \zeta] + 1 - \mu[f > \zeta] \\ &= 1 - \mu[f > \zeta](1 - e^{-\zeta}) \le 1 - \Big(\frac{c_-}{2c}\Big)^2 (1 - e^{-c_-/2}) =: p. \end{aligned} \tag{4.4}$$

□

To apply the lemma we need to prove the existence of constants $0 < c_- < c$ so that for any ϵ, any n and $\tilde z_{t_n}$,

$$c_- \le \mathcal{E}_{t_n,\tilde z_{t_n}}[X], \quad \mathcal{E}_{t_n,\tilde z_{t_n}}[X^2] \le c^2. \tag{4.5}$$

Proof that $\mathcal{E}_{t_n,\tilde z_{t_n}}[X] \ge c_-$.

We claim that under $\mathcal{P}_{t_n,\tilde{z}_{t_n}}$ the time spent at N by the process $(\tilde{z}_t)$ during the time interval $[t_n, t_{n+1}]$ is stochastically larger than the time spent at N during the interval $[0, 2N^2]$ by a simple random walk (x_t) in $\mathbb{Z}$ that starts at time 0 from $\tilde{z}_{t_n}$. Indeed, since $a(N,t) < 1/2$ the intensity with which the process $(\tilde{z}_t)$ jumps from N to $N-1$ is smaller than one, which is the jump rate of (x_t), and so one can construct a coupling of both processes for which $|x_{t-t_n} - N| \geq |\tilde{z}_t - N|$ for all t.

Proof that $\mathcal{E}_{t_n,\tilde{z}_{t_n}}[X^2] \leq c^2$.

Since $\mathcal{E}_{t_n,\tilde{z}_{t_n}}[X^2] \leq \mathcal{E}_{t_n,N}[X^2]$ we just need to prove the inequality when $\tilde{z}_{t_n} = N$. A coupling argument similar to the previous one shows that the time spent at N during $[t_n, t_{n+1}]$ by the random walk $(\tilde{z}_t)$ is stochastically smaller than the time spent at N during $[0, 2N^2]$ by a simple random walk (x_t) which starts in N at time 0 and moves in $[0, N]$, i.e. the jumps to -1 and $N+1$ are suppressed. As in [1], the (x_t) process can be realized as a random walk (y_t) on the whole $\mathbb{Z}$ by identifying sites on $\mathbb{Z}$ modulo repeated reflections around $N+1/2$ and $-1/2$. Calling N_i the images of N under the above reflections we need to bound

$$2\int_0^{t_1} ds \int_s^{t_1} ds' \sum_{i,k} E_N\Big[\mathbf{1}_{y_s=N_i}\mathbf{1}_{y_{s'}=N_k}\Big] \tag{4.6}$$

By the local central limit theorem as in [5] (see also Theorem 3 in [1]) this can be bounded in terms of Gaussian integrals and (4.5) is proved, details are omitted.

5 Proof of (3.8)

As in the previous section we write $\epsilon := N^{-1}$. We use the following notation:

$$\pi(x,t) = P_{z_0,\eta_0}[z_t = x] = \tilde{P}_{z_0}[\tilde{z}_t = x], \quad B(x,t) := (j\epsilon)^{-1} d(x,t)\pi(x,t)$$

so that (3.8) is implied by

$$B(N,t) \geq \delta^* \pi(N,t), \quad t \geq T_2 = \kappa\epsilon^{-2} \tag{5.1}$$

We define

$$T_1 = \epsilon^{-(1-a)}, \quad T_0 = T_1 - \epsilon^{-(1-a)/2}, \quad T_2 = \kappa\epsilon^{-2}, \quad a > 0 \text{ small enough} \tag{5.2}$$

$$p_t(x,y) = \text{transition probability of the simple random walk on } [-N, N] \tag{5.3}$$

(the jumps to $\pm(N+1)$ being suppressed). We postpone the proof of the following three bounds:

- There are $b_1 > 0$ and, for any n, c_n so that

$$B(N,t) \geq b_1 \sum_z p_{T_1}(N,z)\pi(z, t-T_1) - c_n\epsilon^n \tilde{P}_{z_0}[\tilde{z}_{t-T_2} \neq \emptyset] \tag{5.4}$$

- There are $b_2 > 0$, and for any n, c_n so that

$$\pi(N,t) \le b_2 \sum_z p_{T_1}(N,z)\pi(z,t-T_1) + c_n \epsilon^n \tilde{P}_{z_0}[\tilde{z}_{t-T_2} \neq \emptyset] \tag{5.5}$$

- There is $b_3 > 0$ so that

$$\pi(N,t) \ge b_3 \epsilon^3 \tilde{P}_{z_0}[\tilde{z}_{t-T_2} \neq \emptyset] \tag{5.6}$$

Claim: (5.1) follows from (5.4), (5.5), (5.6).
Proof of the Claim:

By (5.6) we get from (5.5)

$$[1 - \frac{c_n}{b_3}\epsilon^{n-3}]\pi(N,t) \le b_2 \sum_z p_{T_1}(N,z)\pi(z,t-T_1) \tag{5.7}$$

and from (5.4)

$$B(N,t) \ge b_1 \sum_z p_{T_1}(N,z)\pi(z,t-T_1) - \frac{c_n}{b_3}\epsilon^{n-3}\pi(N,t) \tag{5.8}$$

By using (5.7) we get from (5.8)

$$B(N,t) \ge \frac{b_1}{b_2}[1 - \frac{c_n}{b_3}\epsilon^{n-3}]\pi(N,t) - \frac{c_n}{b_3}\epsilon^{n-3}\pi(N,t) \tag{5.9}$$

which for a fixed n large enough and all ϵ small enough proves (5.1). $\square$

Proof of (5.4)**.** We need to bound from below $B(N,t) := E_{z_0,\eta_0}\big[\mathbf{1}_{z_t=N}\,\eta_t(N-1)\big]$. We condition on $\mathcal{F}_{t-T_1}$ (the canonical filtration) and denote by $E_{\bar{z},\bar{\eta},t-T_1}$ the conditional expectation given $(\bar{z},\bar{\eta})$, $\bar{z} \neq \emptyset$, the configuration at time $t-T_1$. The realizations where $z_{t-T_1} = \emptyset$ evidently do not contribute to $B(N,t)$.

We denote by $\mathcal{D}$ the event where the births and deaths clocks never ring in the time interval $[t-T_1, t]$ and by $P(\mathcal{D})$ its probability. Then

$$\begin{aligned} E_{\bar{z},\bar{\eta},t-T_1}\Big[\mathbf{1}_{z_t=N}\eta_t(N-1)\Big] &\ge E_{\bar{z},\bar{\eta},t-T_1}\Big[\mathbf{1}_{\mathcal{D}}\;\mathbf{1}_{z_t=N}\eta_t(N-1)\Big] \\ &\ge P[\mathcal{D}]\;\sum_y q_{T_1}(X,(\bar{z},y))\bar{\eta}(y) \end{aligned}$$

where $X = (N, N-1)$, $Y = (y_1, y_2)$ and $q_s(X,Y)$ the probability under the stirring process (SSEP) of going from X to Y in a time s; the second inequality follows because the process conditioned on $\mathcal{D}$ has the law of the stirring process.

Since $P[\mathcal{D}] = e^{-2\epsilon j T_1} = e^{-2\epsilon^a j}$ we have

$$E_{\bar{z},\bar{\eta},t-T_1}\Big[\mathbf{1}_{z_t=N}\,\eta_t(N-1)\Big] \ge e^{-2\epsilon^a j}\;\sum_y q_{T_1}(X,(\bar{z},y))\bar{\eta}(y). \tag{5.10}$$

Writing $Y = (\bar{z}, y)$, $Z = (z_1, z_2)$, $Z^0 = (z_1^0, z_2^0)$, $z_i \in \Lambda_N$, $z_i^0 \in \Lambda_N$, $i = 1, 2$:

$$q_{T_1}(X, Y) = \sum_{Z, Z^0} Q_{T_0}(X, X; Z, Z^0) q_{T_1 - T_0}(Z, Y),$$

where T_0 is defined in (5.2) and Q is the law of the coupling between two stirring $(z_1(s), z_2(s))$ and two independent $(z_1^0(s), z_2^0(s))$ particles as defined in [2]. In particular the coupling is such that $z_1(s) = z_1^0(s)$ for all $s \geq 0$ and moreover given any $\zeta > 0$, for any n there is c_n so that

$$\sum_{Z, Z^0 \in \mathcal{A}^c} Q_{T_0}(X, X; Z, Z^0) \leq c_n \epsilon^n \tag{5.11}$$

where

$$\mathcal{A} = \{(Z, Z^0) : z_1 = z_1^0; |z_2 - z_2^0| \leq \epsilon^{-\frac{1-a}{4} - \zeta}\}. \tag{5.12}$$

Let

$$\mathcal{B} = \{Z^0 : |z_1^0 - z_2^0| \geq \epsilon^{-\frac{1-a}{2} + \zeta}\} \tag{5.13}$$

then

$$q_{T_1}(X, Y) \geq \sum_{Z, Z^0 \in \mathcal{A} \cap \mathcal{B}} Q_{T_0}(X, X; Z, Z^0) q_{T_1 - T_0}(Z, Y). \tag{5.14}$$

We write (see (5.3))

$$\sum_y q_{T_1 - T_0}(Z, (\bar{z}, y)) \bar{\eta}(y) = p_{T_1 - T_0}(z_1, \bar{z}) \sum_y p_{T_1 - T_0}(z_2, y) \bar{\eta}(y) + R(Z). \tag{5.15}$$

$$R(Z) = \sum_y \Big[q_{T_1 - T_0}(Z, (\bar{z}, y)) - p_{T_1 - T_0}(z_1, \bar{z}) p_{T_1 - T_0}(z_2, y) \Big] \bar{\eta}(y)$$

For $(Z, Z^0) \in \mathcal{A} \cap \mathcal{B}$, $Z \in \mathcal{B}' := \{|z_1 - z_2| \geq \frac{1}{2} \epsilon^{-\frac{1-a}{2} + \zeta}\}$. Let

$$\mathcal{C} = \{ \sup_{0 \leq s \leq T_1 - T_0} |z_i(s) - z_i| \leq (T_1 - T_0)^{1/2} \epsilon^{-\zeta}, \, i = 1, 2\}$$

and observe that if $Z \in \mathcal{B}'$ and $Z(\cdot) \in \mathcal{C}$, then (for ϵ, a, ζ small enough)

$$|z_1(s) - z_2(s)| \geq \frac{1}{2} \epsilon^{-\frac{1-a}{2} + \zeta} - 2 \epsilon^{-\frac{1-a}{4} - \zeta} \geq 2, \quad 0 \leq s \leq T_1 - T_0$$

and therefore

$$\mathbb{E}_Z \big[\mathbf{1}_{Z(T_1 - T_0) = Y} \, \mathbf{1}_{\mathcal{C}} \big] = E_Z^0 \big[\mathbf{1}_{Z^0(T_1 - T_0) = Y} \, \mathbf{1}_{\mathcal{C}} \big], \quad Z \in \mathcal{B}'$$

where $\mathbb{E}_Z$ and $\mathbb{E}_Z^0$ denote expectation relative to the stirring and the independent processes, both starting from Z. Since

$$q_{T_1 - T_0}(Z, Y) = \mathbb{E}_Z[\mathbf{1}_{\mathcal{C}} \mathbf{1}_{Z(T_1 - T_0) = Y}] + \mathbb{E}_Z[\mathbf{1}_{\mathcal{C}^c} \mathbf{1}_{Z(T_1 - T_0) = Y}]$$
$$\prod_i p_{T_1 - T_0}(z_i, y_i) = \mathbb{E}_Z^0[\mathbf{1}_{\mathcal{C}} \mathbf{1}_{Z^0(T_1 - T_0) = Y}] + \mathbb{E}_Z^0[\mathbf{1}_{\mathcal{C}^c} \mathbf{1}_{Z^0(T_1 - T_0) = Y}]$$

then for $Z \in \mathcal{B}'$:

$$R(Z) \le \sum_Y \Big(\mathbb{E}_Z[\mathbf{1}_{\mathcal{C}^c}\mathbf{1}_{Z(T_1-T_0)=Y}] + \mathbb{E}^0_Z[\mathbf{1}_{\mathcal{C}^c}\mathbf{1}_{Z^0(T_1-T_0)=Y}]\Big).$$

One also has

$$\mathbb{P}[\mathcal{C}^c] \le 2\sup_z P^0[\sup_{0\le s\le T_1-T_0} |z(s)-z| > (T_1-T_0)^{1/2}\epsilon^{-\zeta}] \le c_n\epsilon^n.$$

The same bound holds for $\mathbb{P}^0[\mathcal{C}^c]$, so that

$$|R(Z)| \le 2c_n\epsilon^n. \tag{5.16}$$

From (5.10), (5.14) and (5.16) we then get

$$\begin{aligned} E_{\bar z,\bar\eta,t-T_1}\Big[\mathbf{1}_{z_t=N}\eta(N-1,t)\Big] &\ge e^{-2\epsilon^a j}\sum_{Z,Z^0\in\mathcal{A}\cap\mathcal{B}} Q_{T_0}(X,X;Z,Z^0) \\ &\times p_{T_1-T_0}(z_1,\bar z)\sum_{y\neq\bar z} p_{T_1-T_0}(z_2,y)\bar\eta(y) - c'_n\epsilon^n. \end{aligned} \tag{5.17}$$

Let

$$\mathcal{G} = \{(z_{t-T_1},\eta_{t-T_1}) : z_{t-T_1}\neq\emptyset,\ \inf_x \sum_{y\neq z_{t-T_1}} p_{T_1-T_0}(x,y)\eta_{t-T_1}(y) \ge \delta^*\}. \tag{5.18}$$

Then for $\tilde z \neq \emptyset$,

$$\begin{aligned} &E_{\bar z,\bar\eta,t-T_1}\Big[\mathbf{1}_{z_t=N}\eta(N-1,t)\Big] \ge e^{-2\epsilon^a j}\delta^*\mathbf{1}_{(\bar z,\bar\eta)\in\mathcal{G}} \\ &\times \sum_{Z,Z^0\in\mathcal{A}\cap\mathcal{B}} Q_{T_0}(X,X;Z,Z^0)p_{T_1-T_0}(z_1,\bar z) - c'_n\epsilon^n. \end{aligned} \tag{5.19}$$

Writing $\mathcal{A}\cap\mathcal{B} = \mathcal{B}\setminus(\mathcal{A}^c\cap\mathcal{B})$,

$$\begin{aligned} \sum_{Z,Z^0\in\mathcal{A}\cap\mathcal{B}} Q_{T_0}(X,X;Z,Z^0)p_{T_1-T_0}(z_1,\bar z) &\ge -Q_{T_0}[\mathcal{A}^c] \\ &+ \sum_{|z^0_1-z^0_2|\ge\epsilon^{-\frac{1-a}{2}+\zeta}} p_{T_0}(N;z^0_1)p_{T_0}(N-1;z^0_2)p_{T_1-T_0}(z^0_1,\bar z). \end{aligned}$$

For any z^0_1

$$\sum_{z^0_2:|z^0_1-z^0_2|\ge\epsilon^{-(1-a)/2+\zeta}} p_{T_0}(N-1,z^0_2) \ge \frac{1}{2},$$

so that by (5.11)

$$\sum_{Z,Z^0\in\mathcal{A}\cap\mathcal{B}} Q_{T_0}(X,X;Z,Z^0)p_{T_1-T_0}(z_1,\bar z) \ge -c_n\epsilon^n + \frac{1}{2}p_{T_1}(N,\bar z).$$

Then taking the expectation in (5.19) we have

$$\begin{aligned} B(N,t) \qquad &\ge e^{-2\epsilon^a j}\frac{\delta^*}{2}\sum_{z\neq\emptyset} p_{T_1}(N,z)\pi(z,t-T_1) \\ &-\big(c_n\epsilon^n P_{z_0,\eta_0}[z_{t-T_1}\neq\emptyset] + e^{-2\epsilon^a j}\delta^* P_{z_0,\eta_0}[\mathcal{G}^c\cap\{z_{t-T_1}\neq\emptyset\}]\big). \end{aligned}$$

In Section 6 we shall prove that

$$P_{z_0,\eta_0}[\mathcal{G}^c\cap\{z_{t-T_1}\neq\emptyset\}] \le c_n\epsilon^n P_{z_0,\eta_0}[z_{t-T_2}\neq\emptyset] \tag{5.20}$$

which will then complete the proof of (5.4). □

Proof of (5.5)**.** (The proof given below uses that the cardinality K of $I_\pm$ is 2, for $K>2$ the proof is similar but more complex). By conditioning on $\tilde z_{t-T_1}$ we get

$$P_{z_0,\eta_0}\big[z_t=N\big] = \tilde P_{z_0}\big[\tilde z_t = N\big] = \tilde E_{z_0}\Big[\mathbf{1}_{\tilde z_{t-T_1}\neq\emptyset}\tilde P_{t-T_1,\tilde z_{t-T_1}}\big[\tilde z_t=N\big]\Big] \tag{5.21}$$

where $\tilde P_{t-T_1,z'}$ is the law of the auxiliary Markov process $\tilde z_s$, $s\ge t-T_1$ which starts at time $t-T_1$ from $z'\neq\emptyset$. Denoting as before by $\mathcal{P}$ and $\mathcal{E}$ the law and expectation of the auxiliary process with generator $\mathcal{L}_t$, i.e. when the death part of the generator is dropped, we have by (3.7),

$$\tilde P_{t-T_1,z'}\big[\tilde z_t=N\big] \le \mathcal{P}_{t-T_1,z'}\big[\tilde z_t=N\big] \tag{5.22}$$

By duality

$$\mathcal{P}_{t-T_1,z'}\big[\tilde z_t=N\big] \le p_{T_1}(N,z') + \int_{t-T_1}^t p_{t-s}(N,N-1)\frac{\epsilon j}{2}\mathcal{P}_{t-T_1,z'}\big[\tilde z_s=N\big] + c_k\epsilon^k$$

with $c_k\epsilon^k$ bounding the contribution of jumps in I_-. We have used that the rate of the extra jumps is $\le \epsilon j/2$, see (3.4).

Iterating

$$\begin{aligned} \mathcal{P}_{z',t-T_1}\big[z_{T_1}=N\big] \le \sum_{n=0}^\infty (\frac{\epsilon j}{2})^n \int_{t-T_1}^t ds_1 \int_{t-T_1}^{s_1} ds_2 \dots \int_{t-T_1}^{s_{n-1}} ds_n \\ p_{t-s_1}(N,N-1)p_{s_1-s_2}(N,N-1)\dots\Big(p_{s_n-(t-T_1)}(N,z') + c_k\epsilon^k\Big). \end{aligned}$$

We write the n-th term of the series as $R_n + R'_n$ where R_n is the term with $s_n \le t-1$ and R'_n the one with $s_n > t-1$. We start by bounding R'_n. After a change of variables ($s_i \to t - s_i$), calling $\underline{s} = (s_1, .., s_n)$ and $s_0 \equiv 0$,

$$\begin{aligned} R'_n &:= (\frac{\epsilon j}{2})^n \int_{[0,T_1]^n, s_n<1} \{\prod_{i=1}^n \mathbf{1}_{s_i \ge s_{i-1}} p_{s_i - s_{i-1}}(N, N-1)\} \Big(p_{T_1 - s_n}(N, z') + c_k \epsilon^k\Big) d\underline{s} \\ &\le (\frac{\epsilon j}{2})^n \int_{[0,1]^n} \{\prod_{i=1}^n \mathbf{1}_{s_i \ge s_{i-1}}\} \Big(p_{T_1 - s_n}(N, z') + c_k \epsilon^k\Big) d\underline{s} \\ &\le \frac{1}{n!} (\frac{\epsilon j}{2})^n \Big(e\, p_{T_1}(N, z') + c_k \epsilon^k\Big). \end{aligned} \tag{5.23}$$

To prove the last inequality we have written

$$p_{T_1 - s_n}(N, z') = \frac{p_{s_n}(N,N)}{p_{s_n}(N,N)} p_{T_1 - s_n}(N, z') \le \frac{p_{T_1}(N, z')}{p_{s_n}(N,N)}$$

and bounded $p_{s_n}(N,N) > e^{-1}$.

To bound R_n we do the same change of variables as above and use the inequality

$$p_{s_i - s_{i-1}}(N, N-1) \le \frac{c}{\sqrt{s_i - s_{i-1}}}$$

Then

$$R_n \le \sum_{n=0}^{\infty} (\frac{\epsilon j}{2})^n \int_{[0,T_1]^n} f(\underline{s}) \Big(p_{T_1 - s_n}(N, z') + c_k \epsilon^k\Big) d\underline{s}$$

where

$$f(\underline{s}) = \mathbf{1}_{0 \equiv s_0 \le s_1 \le s_2 \cdots \le s_n \le T_1} \prod_{i=1}^n \frac{c}{\sqrt{s_i - s_{i-1}}}.$$

Since $p_{s_n}(N,N) > b/\sqrt{s_n}$ (recall that $s_n \ge 1$) getting

$$\begin{aligned} R_n &\le (\frac{\epsilon j}{2})^n \int_{[0,T_1]^n, s_n \ge 1} f(\underline{s}) \Big(\frac{p_{s_n}(N,N)}{p_{s_n}(N,N)} p_{T_1 - s_n}(N, z') + c_k \epsilon^k\Big) d\underline{s} \\ &\le (\frac{\epsilon j}{2})^n \Big(b^{-1} p_{T_1}(N, z') + c_k \epsilon^k\Big) \int_{[0,T_1]^n, s_n \ge 1} f(\underline{s}) \sqrt{s_n} d\underline{s}. \end{aligned}$$

We change variables: $s_i \to T_1 s_i$ and get, using Lemma 5.2 of [1],

$$\begin{aligned} \int_{[0,T_1]^n, s_n \ge 1} f(\underline{s}) \sqrt{s_n} d\underline{s} &\le T_1^{(n+1)/2} \int_{[0,1]^n} f(\underline{s}) \sqrt{s_n} d\underline{s} \\ &\le T_1^{(n+1)/2} \int_{[0,1]^n} f(\underline{s}) d\underline{s} \\ &\le C^n e^{-\frac{n}{2}[\log \frac{n}{2} - 1]} \epsilon^{-\frac{1}{2}(n+1) + \frac{a}{2}(n+1)}. \end{aligned}$$

Thus

$$R_n \le (\frac{Cj}{2})^n e^{-\frac{n}{2}[\log\frac{n}{2}-1]} \epsilon^{\frac{1}{2}(n-1)+\frac{a}{2}(n+1)} \Big(b^{-1} p_{T_1}(N,z') + c_k \epsilon^k \Big). \tag{5.24}$$

The proof of (5.5) follows from (5.21), (5.22), (5.23) and (5.24). □

Proof of (5.6)**.** Let $t \ge T_2 := \kappa\epsilon^{-2}$, then, analogously to (3.7),

$$\pi(N,t) \equiv \tilde P_{z_0}[\tilde z_t = N] = \tilde E_{z_0}\Big[\mathbf{1}_{\tilde z_{t-T_2}\neq\emptyset} \mathcal{E}_{t-T_2,\tilde z_{t-T_2}}\Big[e^{-\int_{t-T_2}^t d(z_s,s)\,ds}; \mathbf{1}_{z_t=N}\Big]\Big] \tag{5.25}$$

with $\mathcal{E}_{t,x}$ as defined in the beginning of Section 4.

We denote by $\mathcal{E}'_N$ the expectation with respect to the time-backward process, z'_s, $s \in [0,T_2]$, which starts at time 0 from N and is a simple random walk with additional jump intensity $a(\pm N, t-s)$ for the jump $\pm(N-1) \to \pm N$ at time s. We then have:

$$\begin{aligned}
\pi(N,t) &= \mathcal{E}'_N\Big[\pi(z'_{T_2}, t-T_2)\exp\{-\int_0^{T_2} d(z'_s, t-s)\,ds\}\Big] \\
&\ge e^{-\epsilon j/2}\mathcal{E}'_N\Big[\pi(z'_{T_2}, t-T_2)\exp\{-\int_1^{T_2-1} d(z'_s, t-s)\,ds\}\Big] \\
&\ge e^{-\epsilon j/2}\mathcal{E}'_N\Big[\pi(z'_{T_2}, t-T_2)\mathbf{1}_{z'_1=N-2}\exp\{-\int_1^{T_2-1} d(z'_s, t-s)\,ds\}\Big] \\
&\ge e^{-\epsilon j/2}\,\alpha \sum_{|x|\le N-2} \pi(x, t-T_2)\ \alpha' E_{N-2}\Big[\mathbf{1}_{x_{T_2-2}=x}\mathbf{1}_{|x_s|<N-1, s\in[0,T_2-2]}\Big] \\
&+ e^{-\epsilon j/2}\,\alpha \sum_{x=\pm\{(N-1),N\}} \alpha'' E_{N-2}\Big[\mathbf{1}_{x_{T_2-2}=\pm(N-2)}\mathbf{1}_{|x_s|<N-1, s\in[0,T_2-2]}\Big]
\end{aligned} \tag{5.26}$$

where E_{N-2} is the expectation of the random walk x_s with no extra jumps and

$$\alpha = \mathcal{P}'_N[z'_1 = N-2] > 0, \quad \alpha' = P_N[x_{T_2-1} = x | x_{T_2-2} = x] > 0,\ |x| < N-1$$

$$\alpha'' = \min_{x=N-1,N} \mathcal{P}'_N[z'_{T_2} = \pm x | z'_{T_2-1} = \pm(N-2)] > 0$$

We thus need to bound from below the probability of the event $\{x_{T_2-2} = x, |x_s| \le N-2, s \in [0, T_2-2]\}$ uniformly in $|x| \le N-2$. The basic idea is to reduce to a single time estimate, indeed the condition $|x_s| \le N-2, s \in [0,T_2-2]$, can be dropped provided we study the process on the whole $\mathbb{Z}$ and take as initial condition the antisymmetric datum which is obtained by assigning a weight ± 1 to the images of x under reflections around $\pm(N-1)$, the details are given in appendix. To have control of the plus and minus contributions it is convenient to reduce to small time intervals, moreover the analysis will distinguish the case where x is "close" to $\pm N$ and when it is not, closeness here means that $N - |x| \le N/100$, (the choice 1/100 is just for the sake of concreteness, any "small" number would work as well).

Let us now be more specific. We split $T_2 - 2 = m\tau\epsilon^{-2}$, m an integer and $\tau > 0$ small enough, and write

$$E_{N-2}\Big[\mathbf{1}_{x_{m\epsilon^{-2}\tau}=x}\mathbf{1}_{|x_s|<N-1, s\in[0,T_2-2]}\Big] \geq$$
$$E_{N-2}\Big[\bigcap_{i=1}^{m-1}\{|x_s| < N-1, s\in[i-1,i]\epsilon^{-2}\tau; |x_{i\epsilon^{-2}\tau}| \leq N/100\}$$
$$\cap\{|x_s| < N-1, s\in[m-1,m]\epsilon^{-2}\tau; x_{m\epsilon^{-2}\tau} = x\}\Big]$$

In an appendix we shall prove that for τ small enough there is c so that for all ϵ ($N = \epsilon^{-1}$) the following bounds hold:

$$E_{N-2}\Big[\mathbf{1}_{|x_{\epsilon^{-2}\tau}|\leq N/100}\mathbf{1}_{|x_s|<N-1, s\in[0,\epsilon^{-2}\tau]}\Big] \geq c\epsilon \tag{5.27}$$

$$\inf_{|x|\leq N/100} E_x\Big[\mathbf{1}_{|x_{\epsilon^{-2}\tau}|\leq N/100}\mathbf{1}_{|x_s|<N-1, s\in[0,\epsilon^{-2}\tau]}\Big] \geq c \tag{5.28}$$

$$\inf_{|x|\leq N/100} E_x\Big[\mathbf{1}_{|x_{\epsilon^{-2}\tau}|\leq N99/100}\mathbf{1}_{|x_s|<N-1, s\in[0,\epsilon^{-2}\tau]}\Big] \geq c \tag{5.29}$$

$$\inf_{|x|\leq N/100}\ \inf_{N99/100\leq|x'|\leq N-2} E_x\Big[\mathbf{1}_{|x_{\epsilon^{-2}\tau}|=x'}\mathbf{1}_{|x_s|<N-1, s\in[0,\epsilon^{-2}\tau]}\Big] \geq c\epsilon^2 \tag{5.30}$$

The above bounds together with (5.26) prove (5.6). □

6 Proof of (5.20)

For any (z,η), we define the configurations $\eta^{(1)}$ and $\eta^{(2)}$ in $\{0,1\}^{\Lambda_N}$ as follows: If $z \neq \emptyset$, then $\eta^{(1)}(x) = \eta^{(2)}(x) = \eta(x)$ for any $x \in \Lambda_N \setminus z$, and $\eta^{(1)}(z) = 1$, $\eta^{(2)}(z) = 0$. If $z = \emptyset$ then $\eta^{(1)} = \eta^{(2)} = \eta$.

If $(z_t, \eta_t)_{t\geq 0}$ is the process defined in Section 2 we can see that $(\eta_t^{(2)})_{t\geq 0}$ has the law of the DPTV process (as well as $(\eta_t^{(1)})_{t\geq 0}$, though such a property will not be used in the following). Details can be found in [4].

For any $x \in \Lambda_N$ we introduce the function $A_x(\eta)$, $\eta \in \{0,1\}^{\Lambda_N}$, by setting

$$A_x(\eta) := \sum_y p_{T_1-T_0}(x,y)\eta(y), \quad \eta \in \{0,1\}^{\Lambda_N}. \tag{6.1}$$

Then, recalling that $\mathcal{G}$ has been defined in (5.18) and writing $\tau := t - T_1$, the left hand side of (5.20) is equal to

$$P_{z_0,\eta_0}\Big[z_\tau \neq \emptyset,\ \inf_x A_x(\eta_\tau^{(2)}) \leq \delta^*\ \Big] \leq E_{z_0,\eta_0}\Big[\mathbf{1}_{z_{t-T_2}\neq\emptyset} P_{z_{t-T_2},\eta_{t-T_2}}[\inf_x A_x(\eta_\tau^{(2)}) \leq \delta^*\ ]\Big]$$

which is bounded by

$$\tilde{P}_{z_0}[\tilde{z}_{t-T_2} \neq \emptyset] \sup_{\eta \in \{0,1\}^{\Lambda_N}} \mathbf{P}_\eta[\inf_x A_x(\eta_S) < \delta^*], \quad S = T_2 - T_1$$

where $\mathbf{P}_\eta$ is the law of the DPTV process starting from η. We thus need to prove that:

$$\sup_{\eta \in \{0,1\}^{\Lambda_N}} \mathbf{P}_\eta[\inf_x A_x(\eta_S) < \delta^*] \leq c_n \epsilon^n, \quad S = T_2 - T_1.$$

Since the evolution preserves the coordinate-wise order in $\{0,1\}^{\Lambda_N}$ (see [2]) and $\inf_x A_x(\eta)$ is a non decreasing function of η, it suffices to show that

$$\mathbf{P_0}[\inf_x A_x(\eta_S) < \delta^*] \leq c_n \epsilon^n, \quad S = T_2 - T_1 \tag{6.2}$$

with $\mathbf{0}$ the configuration with $\eta(x) = 0$ for all x.

In [2] it is proved that there is $\tau^* > 0$ (independent of N) so that if $t \in N^2[1, \tau^* \log N]$ then for any n there is c_n so that:

$$\mathbf{P_0}\Big[\inf_x |A_x(\eta(\cdot,t)) - A_x(\gamma(\cdot,t))| \geq \epsilon^{1/4}\Big] \leq c_n \epsilon^n \tag{6.3}$$

where $\gamma(x,t) = \rho(\epsilon x, \epsilon^2 t)$ and $\rho(r,t)$, $r \in [-1,1]$, $t \geq 0$, is the solution of the hydrodynamic equation for the DPTV system starting from $\rho(r,0) \equiv 0$. In [3] it is proved that

$$\lim_{t \to \infty} \sup_{|r| \leq 1} |\rho(r,t) - \rho^{\rm st}(r)| = 0 \tag{6.4}$$

and that $\rho^{\rm st}(r)$ is an increasing function (linear with positive slope) with $\rho^{\rm st}(-1) > 0$. Thus there is $\kappa > 0$ independent of N so that for all N large enough

$$\mathbf{P}_0\Big[\inf_x A_x(\eta_s) \geq \frac{\rho^{\rm st}(-1)}{2}\Big] \geq 1 - c_n \epsilon^n, \quad \frac{\kappa}{2} N^2 \leq s \leq \kappa N^2 \tag{6.5}$$

Hence (6.3) with $\delta^* < \rho^{\rm st}(-1)/2$ and $T_2 = \kappa N^2$.

7 Appendix

We now prove the bounds (5.27)–(5.30). The key point is the well known identity

$$P_x\Big[|x_t| = y; |x_s| < N-1, s \in [0, \epsilon^{-2}\tau]\Big] = \sum_n (-1)^n p_t(y_n - x) \tag{7.1}$$

where x and y in (7.1) are in $[-(N-2), N-2]$; $\{y_n\}$ are the images of y under reflections around $\pm(N-1)$, n the number of reflections, $y_0 \equiv y$; finally $p_t(y_n - x)$ is the probability that the simple random walk on $\mathbb{Z}$ which starts from x is at y_n at time t.

More explicitly, calling $L = N-1$ the basic interval is $[-L, L]$, we then have $[L, 3L]$, $[3L, 5L]$,.. while on the left we have $[-3L, -L]$, $[-5L, -3L]$,... We attribute pluses and minuses to the above intervals with alternating signs starting from a plus for the basic interval. The images y_n of y have the sign of the interval where they are. Thus calling $z = L - y$, the images y_n of y are: $(4k+1)L - z$, $k \geq 0$, which contribute with a positive sign; $(4k+1)L + z$, $k \geq 0$, which contribute with a negative sign; $-(4k-1)L + z$, $k \geq 1$, which contribute with a negative sign and finally $-(4k-1)L - z$, $k \geq 1$, which contribute with a positive sign. Calling $w = L - x$, (7.1) becomes

$$
\begin{aligned}
&\sum_{k=0}^{\infty} p_t\big((4k+1)L - z - (L-w)\big) - \sum_{k=1}^{\infty} p_t\big((L-w) - [-(4k-1)L + z]\big) \\
&- \sum_{k=0}^{\infty} p_t\big((4k+1)L + z - (L-w)\big) + \sum_{k=1}^{\infty} p_t\big((L-w) - [-(4k-1)L - z]\big) \\
&= \sum_{k=1}^{\infty} [p_t\big(4kL - z + w\big) - p_t\big(4kL - z - w\big)] + p_t\big(w - z\big) \\
&- \sum_{k=1}^{\infty} [p_t\big(4kL + z + w\big) - p_t\big(4kL + z - w\big)] - p_t\big(w + z\big)
\end{aligned}
$$

Thus (7.1) becomes:

$$
\begin{aligned}
&P_x\Big[|x_t| = y; |x_s| < N-1, s \in [0, \epsilon^{-2}\tau]\Big] = p_t(z - w) - p_t(z + w) \qquad (7.2)\\
&\quad + \sum_{k=1}^{\infty} \Big([p_t\big(4kL - z + w\big) - p_t\big(4kL - z - w\big)] - [p_t\big(4kL + z + w\big) - p_t\big(4kL + z - w\big)] \Big)
\end{aligned}
$$

To prove (5.27) (where $x = N-2$) we take $w = 1$ in (7.2) and get

$$
\begin{aligned}
P_{N-2}\Big[|x_t| = y; |x_s| < N-1, s \in [0, t]\Big] &\geq p_t(z-1) - p_t(z+1) \qquad (7.3)\\
&- \sum_{1 \leq k \leq \epsilon^{-b}} \sum_{\sigma = \pm 1} \Big(|p_t(4kL + \sigma z - 1) - p_t(4kL + \sigma z + 1)| \Big) \\
&-2 \sum_{|y| \geq N\epsilon^{-b}/2} p_t(y), \qquad z = L - y,\ L = N - 1
\end{aligned}
$$

$b > 0$ a small constant. We shall use the smallness of τ to prove that the sum over $1 \leq k \leq \epsilon^{-b}$ is a small fraction of the first term. Moreover, there is $c > 0$ so that for all ϵ small enough

$$
\sum_{|y| \geq N\epsilon^{-b}/2} p_t(y) \leq e^{-c\epsilon^{-2b}} \qquad (7.4)
$$

as the left hand side is the probability that a random walk goes past $\pm\epsilon^{-1-b}$ in a time $\epsilon^{-2}\tau$ (b and τ positive constants independent of ϵ). We shall prove that $p_t(z-1) - p_t(z+1)$ is

bounded from below proportionally to ϵ^2, so that the last sum in (7.2) will be negligible. The other terms on the right hand side of (7.2) are bounded in the following proposition:

Proposition 3. *Recalling that $N \equiv \epsilon^{-1}$, $t \equiv \epsilon^{-2}\tau$ there are positive constants c, C and b such that for every τ, the following holds for all ϵ small enough:*

- *When $N/2 < y < 2N$,*

$$p_t(y) - p_t(y+2) \geq \frac{\epsilon^2}{\sqrt{2\pi\tau}} e^{-(\epsilon y)^2/2\tau} \frac{1}{4\tau}(1 - c\epsilon) \tag{7.5}$$

- *When $N/2 < y < N\epsilon^{-b}$,*

$$p_t(y) - p_t(y+2) \leq \frac{\epsilon^2}{\sqrt{2\pi\tau}} e^{-(\epsilon y)^2/2\tau} \frac{8\epsilon y}{\tau}(1 + c\epsilon) \tag{7.6}$$

Proof. We have

$$p_t(y) = e^{-t} \sum_n{}^{*} (\frac{1}{2})^n \frac{t^n}{n!} \binom{n}{m}, \quad y = 2m - n$$

where $\sum_n{}^*$ means that n runs over either the odd or the even integers of $\mathbb{Z}$ according to whether y is odd or respectively even. n is the total number of jumps, m the number of jumps to the right so that $m - (n - m) = y$.

We start by proving (7.5). For every pair y and $y' := y+2$ let m and m' be the number of the corresponding jumps to the right, so that $m' = m + 1$. Then

$$\binom{n}{m} - \binom{n}{m'} = \binom{n}{m}(1 - \frac{n-m}{m+1}) = \binom{n}{m}\frac{y+1}{m+1} \tag{7.7}$$

We bound $m = (n + y)/2 \leq t$, which is valid when $n \leq 2t - 2N$. Thus

$$p_t(y) - p_t(y+2) \geq \frac{N}{2(t+1)} e^{-t} \sum_{n \leq 2t-2N} (\frac{1}{2})^n \frac{t^n}{n!} \binom{n}{m}$$

(7.5) then follows from the local limit theorem, [5], after observing that the sum over $n > 2t - 2N$ is exponentially small in t.

To prove (7.6) we proceed similarly. Since we want an upper bound, we write $m + 1 \geq n/2$, getting

$$p_t(y) - p_t(y+2) \leq \frac{y+1}{t/4} e^{-t} \sum_{n \geq t/2} (\frac{1}{2})^n \frac{t^n}{n!} \binom{n}{m}$$

As before (7.6) is then a consequence of the local theorem and the large deviation estimate for the set $n < t/2$.

$\square$

Proof of (5.27)**.** By (7.3) using the above proposition:

$$
\begin{aligned}
P_{N-2}\Big[|x_t| = y; |x_s| < N-1, s \in [0,t]\Big] &\ge \frac{\epsilon^2}{\sqrt{2\pi\tau}} e^{-(\epsilon z)^2/2\tau} \frac{1}{4\tau}(1-c\epsilon)\\
&-2 \sum_{1\le k\le \epsilon^{-b}} \frac{\epsilon^2}{\sqrt{2\pi\tau}} e^{-([4k(1-\epsilon)-\epsilon z-\epsilon])^2/2\tau} \frac{8(4k+2)}{\tau}(1+c\epsilon)\\
&-2e^{-c\epsilon^{-2b}}, \qquad z = N-1-y
\end{aligned}
$$

If $\tau > 0$ is sufficiently small then for all ϵ small enough

$$
P_{N-2}\Big[|x_t| = y; |x_s| < N-1, s \in [0,t]\Big] \ge \frac{\epsilon^2}{\sqrt{2\pi\tau}} e^{-(\epsilon z)^2/2\tau} \frac{1}{8\tau}
$$

and (5.27) is proved.

To prove (5.28) and (5.29) we use again (7.1) and bound

$$
\sum_{n\in\mathbb{Z}} (-1)^n p_t(y_n - x) \ge p_t(y_0 - x) - \sum_{|n|\ge 1} p_t(y_n - x) \tag{7.8}
$$

using that successive images y_n have mutual distance $\ge aN$, a some positive constant. As before we bound the right hand side by

$$
p_t(y_0 - x) - \sum_{1\le |n| \le N\epsilon^{-b}} p_t(y_n - x) - \sum_{|z|\ge N\epsilon^{-b}} p_t(z)
$$

and (5.28) and (5.29) follow using the local theorem and large deviations as before.

Proof of (5.30). We use the equality

$$
E_x\Big[\mathbf{1}_{|x_{\epsilon^{-2}\tau}|=x'} \mathbf{1}_{|x_s|<N-1, s\in[0,\epsilon^{-2}\tau]}\Big] = E_{x'}\Big[\mathbf{1}_{|x_{\epsilon^{-2}\tau}|=x} \mathbf{1}_{|x_s|<N-1, s\in[0,\epsilon^{-2}\tau]}\Big] \tag{7.9}
$$

recalling that $|x| \le N/100$ and $N99/100 \le |x'| \le N-2$; we thus need to bound the right hand side of (7.9) by $c\epsilon^2$ with $c > 0$ independent of x and x' when they vary in the above sets.

We thus use (7.2) with $x \to x'$ and $y \to x$, so that, on the right hand side we must read $z = L - x$ and $w = L - x'$. Observe that $z \le N - 1 - N/100$ and $w \in [N-1-N/100, N-1+N/100]$. To have the same structure as in (7.3) we write

$$
\begin{aligned}
p_t(z-w) - p_t(z+w) \quad &= \quad [p_t([z-w+1]-1) - p_t([z-w+1]+1)] + \cdots\\
&\cdots \quad +[p_t([z+w-1]-1) - p_t([z+w-1]+1)]
\end{aligned}
$$

with the analogous decomposition for $p_t(z'-w) - p_t(z'+w)$ with $z' = 4kL + \pm z$ Call Y the set of all y of the form $y = z - w + (2n+1)$, $n \le \bar{n}$ where $z - w + (2\bar{n}+1) = z + w - 1$,

then

$$P_{x'}\Big[|x_t| = x; |x_s| < N-1, s \in [0, \epsilon^{-2}\tau]\Big] \ge \sum_{y\in Y}\Big([p_t(z-y-1) - p_t(z-y+1)]$$
$$- \sum_{1\le k\le N\epsilon^{-b}} \sum_{\sigma=\pm 1} |p_t\big(4kL - \sigma(z-y) - 1\big) - p_t\big(4kL - \sigma(z-y)+1\big)|$$
$$-2 \sum_{|x|\ge N\epsilon^{-b}/2} p_t(x)\Big) \tag{7.10}$$

and for each y we have the same bound as before, hence (5.30).

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